Spectral Properties of Fractional Sturm-Liouville Problem for Diffusion Operator

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Abstract. In this study, we give a regular fractional Sturm Liouville problem for diffusion operator (FSLPDO), research the spectral properties of the eigenfunctions and eigenvalues of the diffusion operator. We show that the eigenvalues and eigenfunctions of (FSLPDO) are real and orthogonal, respectively and fractional diffusion operator is self adjoint.

Keywords. Sturm-Liouville, Fractional, Diffusion Operator, Spectral Properties.

AMS subject classifications: 26A33, 34A08

1. Introduction

Mathematical techniques could be developed into a more suitable and significant course by presenting them within the more general Sturm-Liouville theory in L_2. The Sturm-Liouville Problems are important in many areas of science, engineering and mathematics. It is known that the spectral characteristics are spectra, spectral functions, scattering data, norming constants, etc. According to the theory a linear second-order differential operator which is self-adjoint has an orthogonal sequence of eigenfunctions in L_2. Spectral properties of Sturm-Liouville operators are often derived, directly or indirectly, as a consequence of an established link between large distance asymptotic behavior of solutions of the associated differential equation and spectral properties of the corresponding differential operator. Sturm-Liouville problems are divided into regular and singular type. Differential equations such as Bessel, hydrogen atom, Hermitte, Jakobi, and Legendre equations can be transformed into Sturm Liouville equations. There are many studies on these issues, $[1-5]$.

Fractional calculus is "the theory of derivatives and integrals of any arbitrary real or complex order, which unify and generalize the notions of integer-order differentiation and $n-$fold integration" $[6-13]$. In recent years, fractional calculus has been of great interest and there has been many results on the topic. It has been proved that many systems in different fields of science and engineering can be modeled more accurately using fractional derivatives $[14-15]$. We note that ordinary derivatives in a traditional Sturm-Liouville problem are replaced with fractional derivatives and the resulting problems are solved using some numerical methods [16, 17]. Furthermore, Klimek and Agrawal [18] define a fractional Sturm-Liouville operator, introduce a regular FSLP and investigate the properties of the eigenfunctions and the eigenvalues of the operator. In this paper, our purpose is to give regular fractional Sturm-Liouville problem for diffusion operator (FSLPDO) and prove fundamental properties of spectral data for the operator.

2. Preliminaries

Definition 1. Let $\operatorname{Re}(\alpha) > 0$. The left-sided and respectively right-sided Riemann-Liouville integrals of order α are given by the formulas

$$\left(I_{a,+}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-s)^{\alpha-1} f(s)\, ds, \quad x > a \tag{1}$$

$$\left(I_{b,-}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (s-x)^{\alpha-1} f(s)\, ds, \quad x < b \tag{2}$$

where Γ denotes the gamma function.

Definition 2. Let $\mathrm{Re}(\alpha) > 0$. The left-sided and respectively right-sided Riemann-Liouville derivatives of order α are defined as

$$\left(D_{a,+}^{\alpha} f\right)(x) = D\left(I_{a,+}^{1-\alpha} f\right)(x) \qquad x > a \tag{3}$$

$$\left(D_{b,-}^{\alpha} f\right)(x) = -D\left(I_{b,-}^{1-\alpha} f\right)(x) \qquad x < b \tag{4}$$

Analogous formulas yield the left- and right-sided Caputo derivatives of order α:

$$\left({}^{C}D_{a,+}^{\alpha} f\right)(x) = \left(I_{a,+}^{1-\alpha} Df\right)(x) \qquad x > a \tag{5}$$

$$\left({}^{C}D_{b,-}^{\alpha} f\right)(x) = \left(I_{b,-}^{1-\alpha} (-D) f\right)(x) \qquad x < b \tag{6}$$

Definition 3.[13] The general function ${}_p\Psi_q(z)$ is defined for $z \in \mathbb{C}$, complex $a_l, b_j \in \mathbb{C}$, and real $\alpha_l, \beta_j \in \mathbb{R}\,(l = 1, ..., p; j = 1, ...q)$ by the series

$${}_p\Psi_q(z) = {}_p\Psi_q\left[\begin{array}{c} (a_1, \alpha_1)_{1,p} \\ (b_1, \beta_1)_{1,q} \end{array} \middle| z\right] = \sum_{k=0}^{\infty} \frac{\prod_{l=1}^{p} \Gamma(a_l + \alpha_l k)}{\prod_{j=1}^{q} \Gamma\left(b_j + \beta_j k\right)} \frac{z^k}{k!} \tag{7}$$

where $z, a_l, b_j \in \mathbb{C}, \alpha_l, \beta_j \in \mathbb{R}\,(l = 1, ..., p; j = 1, ...q)$. This general Wright function was investigated by Fox who presented its asymptotic expansion for large values of the argument z under the condition

$$\sum_{j=1}^{q} \beta_j - \sum_{l=1}^{p} \alpha_l > 1.$$

If these conditions are satisfied, the series in (7) is convergent for any $z \in \mathbb{C}$.

Teorem 4. [13] Let $a_l, b_j \in \mathbb{C}$, and $\alpha_l, \beta_j \in \mathbb{R}\,(l = 1, ..., p; j = 1, ...q)$ and let

$$\begin{aligned} \Delta &= \sum_{j=1}^{q} \beta_j - \sum_{l=1}^{p} \alpha_l, \\ \delta &= \prod_{l=1}^{p} |\alpha_l|^{-\alpha_l} \prod_{j=1}^{q} \left|\beta_j\right|^{\beta_j}, \\ \mu &= \sum_{j=1}^{q} b_j - \sum_{l=1}^{p} a_l + \frac{p-q}{2} \end{aligned}$$

I) If $\Delta > -1$, then the series in (7) is absolutely convergent for all $z \in \mathbb{C}$.

II) If $\Delta > -1$, then the series in (7) is absolutely convergent for $|z| < \delta$ and for $|z| = \delta$ and $\Re(\mu) > 1/2$.

Definition 5.[19] Let (X, d) be a non-empty complete metric space. Let $T : X \to X$ be a contraction mapping on X, i.e.: there is a nonnegative real number $q < 1$ such that

$$d(T(x), T(y)) \leq q d(x, y)$$

for all x, y in X. Then the map T admits one and only one fixed-point x^* in X (this means $T(x^*) = x^*$). Furthermore, this fixed point can be found as follows start with an arbitrary element

x_0 in X and define an iterative sequence by $x_n = T(x_{n-1})$ for $n = 1, 2, 3, ...$ This sequence converges, and its limit is x^*. The following inequality describes the speed of convergence:

$$d(x^*, x_n) \le \frac{q_n}{1-q} d(x_1, x_0).$$

Equivalently,

$$d(x^*, x_{n+1}) \le \frac{q}{1-q} d(x_{n+1}, x_n)$$

and

$$d(x^*, x_{n+1}) \le q d(x^*, x_n).$$

Any such value of q is called a Lipschitz constant for T, and the smallest one is sometimes called "the best Lipschitz constant" of T.

Property 6. The fractional differential operators defined in $(3-6)$ satisfy the following identities:

$$\int_a^b f(x) D_{b,-}^{\alpha} g(x) dx = \int_a^b g(x)^C D_{a,+}^{\alpha} f(x) dx - f(x) I_{b,-}^{1-\alpha} g(x) |_a^b$$

or

$$\begin{aligned} &\int_a^b f(x) D_{b,-}^{\alpha} g(x)^C D_{a,+}^{\alpha} k(x) dx \\ = &\int_a^b g(x)^C D_{a,+}^{\alpha} f(x)^C D_{a,+}^{\alpha} k(x) dx - f(x) I_{b,-}^{1-\alpha} g(x)^C D_{a,+}^{\alpha} k(x) |_a^b, \end{aligned} \tag{8}$$

$$\int_a^b f(x) D_{a,+}^{\alpha} g(x) dx = \int_a^b g(x)^C D_{b,-}^{\alpha} f(x) dx + f(x) I_{a,+}^{1-\alpha} g(x) |_a^b. \tag{9}$$

Property 7. Assume $\alpha \in (0,1)$, $\beta > \alpha$ and $f \in C[a,b]$. Then the following relations

$$\begin{aligned} D_{a,+}^{\alpha} I_{a,+}^{\alpha} f(x) &= f(x) \\ D_{b,-}^{\alpha} I_{b,-}^{\alpha} f(x) &= f(x) \\ D_{a,+}^{\alpha} I_{a,+}^{\beta} f(x) &= I_{a,+}^{\beta-\alpha} f(x) \\ D_{b,-}^{\alpha} I_{b,-}^{\beta} f(x) &= I_{b,-}^{\beta-\alpha} f(x) \\ {}^C D_{a,+}^{\alpha} I_{a,+}^{\alpha} f(x) &= f(x) \\ {}^C D_{b,-}^{\alpha} I_{b,-}^{\alpha} f(x) &= f(x) \end{aligned} \tag{10}$$

hold for any $x \in [a,b]$. Furthermore, the integral operators defined in $(1,2)$ satisfy the following semi-group properties,

$$I_{a,+}^{\alpha} I_{a,+}^{\beta} = I_{a,+}^{\alpha+\beta}, \qquad I_{b,-}^{\alpha} I_{b,-}^{\beta} = I_{b,-}^{\alpha+\beta}$$

Now, let's take up a fractional Sturm-Liouville problem for diffusion operator

3. Main Results

A Regular Fractional Sturm-Liouville Problem for Diffusion Operator

Let's denote a fractional Sturm-Liouville problem for diffusion operator with the differential part containing the left-and right-sided derivatives. Let's use the form of the integration by parts formulas $(8, 9)$ for this new approximation. Main properties of eigenfunctions and eigenvalues in the theory of classical Sturm-Liouville problems are related to the integration by parts formula for the first order derivatives. In the corresponding fractional version we note that both left and right derivatives appear and the essential pairs are the left Riemann-Liouville derivative with the right Caputo derivative and the right Riemann-Liouville derivative with the left Caputo one.

Definition 8. Let $\alpha \in (0,1)$. Fractional diffusion operator is written as

$$L_{[\alpha,p,q]} = D^{\alpha}_{\pi,-} h(x)^{C} D^{\alpha}_{0,+} + (2\lambda p(x) + q(x)) \tag{11}$$

where the function $q(x) \in L^1[0,\pi], p(x) \in L^2[0,\pi]$. Consider the fractional diffusion equation

$$L_{[\alpha,p,q]} y_\lambda(x) + \lambda w_\alpha(x) y_\lambda(x) = 0 \tag{12}$$

where $h(x), p(x) \neq 0, w_\alpha(x) > 0\ \forall x \in [0,\pi]$ and p, q, h, w_α are real valued continuous functions in interval $[0,\pi]$. The boundary conditions for the operator L are the following:

$$c_1 y_\lambda(x) - c_2 I^{1-\alpha}_{\pi,-} h(0)^{C} D^{\alpha}_{0,+} y(x)\Big|_{x=0} = 0 \tag{13}$$

$$d_1 y(\pi) + d_2 I^{1-\alpha}_{\pi,-} h(\pi)^{C} D^{\alpha}_{0,+} y(\pi) = 0 \tag{14}$$

where $d_1^2 + d_2^2 \neq 0, c_1^2 + c_2^2 \neq 0$. The fractional boundary-value problem (12)-(14) is fractional Sturm-Liouville problem for diffusion operator FSLPDO.

Teorem 9. Fractional diffusion operator is self-adjoint on $[0,\pi]$.

Proof: Let us consider the following equation

$$\begin{aligned} \left\langle L_{[\alpha,p,q]}\varphi, \phi \right\rangle &= \int_0^\pi L_{[\alpha,p,q]}\varphi(x) . \phi(x)\, dx = \int_0^\pi \phi(x) D^{\alpha}_{\pi,-} h(x)^{C} D^{\alpha}_{0,+}\varphi(x)\, dx \\ &+ \int_0^\pi (2\lambda p(x) + q(x))\, \varphi(x)\, \phi(x)\, dx. \end{aligned}$$

By means of property (8) and boundary conditions $(13-14)$ and we obtain the identity

$$\begin{aligned} \left\langle L_{[\alpha,p,q]}\varphi, \phi \right\rangle &= \int_0^\pi h(x)^{C} D^{\alpha}_{0,+}\phi(x)^{C} D^{\alpha}_{0,+}\varphi(x)\, dx + \frac{d_1}{d_2}\varphi(\pi)\phi(\pi) + \frac{c_1}{c_2}\varphi(0)\phi(0) \\ &+ \int_0^\pi (2\lambda p(x) + q(x))\, \varphi(x)\, \phi(x)\, dx. \end{aligned} \tag{15}$$

On the other hand,

$$\begin{aligned} \left\langle \varphi, L_{[\alpha,p,q]}\phi \right\rangle &= \int_0^\pi \varphi(x) L_{[\alpha,p,q]}\phi(x)\, dx = \int_0^\pi \varphi(x) D^{\alpha}_{\pi,-} h(x)^{C} D^{\alpha}_{0,+}\phi(x)\, dx \\ &+ \int_0^\pi \varphi(x) (2\lambda p(x) + q(x))\, \phi(x)\, dx. \end{aligned}$$

Similarly, we obtain

$$\begin{aligned}\left\langle \varphi, L_{[\alpha,p,q]}\phi\right\rangle &= \int_0^\pi h(x)^C D_{0,+}^{\alpha}\varphi(x)^C D_{0,+}^{\alpha}\phi(x)\,dx + \frac{d_1}{d_2}\varphi(\pi)\phi(\pi) + \frac{c_1}{c_2}\varphi(0)\phi(0) \\ &+ \int_0^\pi (2\lambda p(x) + q(x))\phi(x)\varphi(x)\,dx. \qquad (16)\end{aligned}$$

The right hand sides of the equations (15) and (16) are equal hence we may see that the left sides are equal that is

$$\left\langle L_{[\alpha,p,q]}\varphi, \phi\right\rangle = \left\langle \varphi, L_{[\alpha,p,q]}\phi\right\rangle.$$

Therefore, $L_{[\alpha,p,q]} = L^*_{[\alpha,p,q]}$. The proof is completed.

Teorem 10. The eigenvalues of FSLPDO $(12-14)$ are real.

Proof: Let us observe that following relation results from Property (8)

$$\begin{aligned}\int_0^\pi f(x) L_{[\alpha,p,q]}g(x)\,dx &= \int_a^b h(x)^C D_{0,+}^{\alpha} f(x)^C D_{0,+}^{\alpha} g(x)\,dx - f(x) I_{\pi,-}^{1-\alpha} h(x)^C D_{0,+}^{\alpha} g(x)\mid_0^\pi \\ &+ \int_0^\pi (2\lambda p(x) + q(x)) g(x) f(x)\,dx. \qquad (17)\end{aligned}$$

Suppose that λ is the eigenvalue for $(12-14)$ corresponding to eigenfunction y the following equalities are satisfy y and its complex conjugate $\bar{y}$

$$L_{[\alpha,p,q]}y(x) + \lambda w_\alpha(x) y(x) = 0 \qquad (18)$$

$$c_1 y_\lambda(x) - c_2 I_{\pi,-}^{1-\alpha} h(0)^C D_{0,+}^{\alpha} y(x)\Big|_{x=0} = 0 \qquad (19)$$

$$d_1 y(\pi) + d_2 I_{\pi,-}^{1-\alpha} h(\pi)^C D_{0,+}^{\alpha} y(\pi) = 0 \qquad (20)$$

$$L_{[\alpha,p,q]}\bar{y}(x) + \bar{\lambda} w_\alpha(x)\bar{y}(x) = 0 \qquad (21)$$

$$c_1 \bar{y}(x) - c_2 I_{\pi,-}^{1-\alpha} h(0)^C D_{0,+}^{\alpha} \bar{y}(x)\Big|_{x=0} = 0 \qquad (22)$$

$$d_1 \bar{y}(\pi) + d_2 I_{\pi,-}^{1-\alpha} h(\pi)^C D_{0,+}^{\alpha} \bar{y}(\pi) = 0 \qquad (23)$$

where $d_1^2 + d_2^2 \neq 0, c_1^2 + c_2^2 \neq 0$. We multiply equation (18) by function $\bar{y}$ and (21) by function y respectively and subtract:

$$\left(\lambda - \bar{\lambda}\right) w_\alpha(x) y(x)\bar{y}(x) = y(x) L_{[\alpha,p,q]}\bar{y}(x) - \bar{y}(x) L_{[\alpha,p,q]}y(x).$$

Now, we integrate over interval $[0,\pi]$ and applying relation (17) we note that the right-hand side

of the integrated equality contains only boundary terms:

$$
\begin{aligned}
\left(\lambda-\bar{\lambda}\right)\int_0^\pi w_\alpha(x)\,y(x)\,\bar{y}(x)\,dx &= \int_0^\pi y(x)\,L_{[\alpha,p,q]}\bar{y}(x)\,dx-\int_0^\pi \bar{y}(x)\,L_{[\alpha,p,q]}y(x)\,dx\\
&= \int_0^\pi 2\bar{\lambda}p(x)\,y(x)\,\bar{y}(x)\,dx-\int_0^\pi 2\lambda p(x)\,y(x)\,\bar{y}(x)\,dx\\
-y(x)\,I_{\pi,-}^{1-\alpha}h(x)^C\,D_{0,+}^{\alpha}\bar{y}(x) &\mid_0^\pi+\bar{y}(x)\,I_{\pi,-}^{1-\alpha}h(x)^C\,D_{0,+}^{\alpha}y(x)\mid_0^\pi\\
\left(\lambda-\bar{\lambda}\right)\int_0^\pi \left(w_\alpha(x)+2p(x)\right)|y(x)|^2 &= -y(x)\,I_{\pi,-}^{1-\alpha}h(x)^C\,D_{0,+}^{\alpha}\bar{y}(x)\mid_0^\pi+\bar{y}(x)\,I_{\pi,-}^{1-\alpha}h(x)^C\,D_{0,+}^{\alpha}y(x)\mid_0^\pi,
\end{aligned}
$$

by virtue of the boundary conditions $(19),(20),(22),(23)$, we find

$$
\left(\lambda-\bar{\lambda}\right)\int_0^\pi \left(w_\alpha(x)+2p(x)\right)|y(x)|^2\,dx=0
$$

and because y is a non-trivial solution and $p(x)\neq 0, w_\alpha(x)>0$, it easily seen that $\lambda=\bar{\lambda}$. This proves the theorem.

Teorem 11. The eigenfunctions corresponding to distinct eigenvalues of FSLPDO $(12-14)$ are orthogonal weight function w_α on $[0,\pi]$ that is

$$
\int_0^\pi \left(w_\alpha(x)+2p(x)\right)y_{\lambda_1}(x)\,y_{\lambda_2}(x)\,dx=0 \qquad \lambda_1\neq\lambda_2.
$$

Proof. We have by assumptions fractional Sturm-Liouville for diffusion operator fulfilled by two different eigenvalues (λ_1,λ_2) and the respective eigenfunctions $(y_{\lambda_1},y_{\lambda_2})$:

$$L_{[\alpha,p,q]}y_{\lambda_1}(x)+\lambda_1 w_\alpha(x)\,y_{\lambda_1}(x)=0 \tag{24}$$

$$c_1 y_{\lambda_1}(x)-\left.c_2 I_{\pi,-}^{1-\alpha}h(0)^C\,D_{0,+}^{\alpha}y_{\lambda_1}(x)\right|_{x=0}=0 \tag{25}$$

$$d_1 y_{\lambda_1}(\pi)+d_2 I_{\pi,-}^{1-\alpha}h(\pi)^C\,D_{0,+}^{\alpha}y_{\lambda_1}(\pi)=0 \tag{26}$$

$$L_{[\alpha,p,q]}y_{\lambda_2}(x)+\lambda_2 w_\alpha(x)\,y_{\lambda_2}(x)=0 \tag{27}$$

$$c_1 y_{\lambda_2}(x)-\left.c_2 I_{\pi,-}^{1-\alpha}h(0)^C\,D_{0,+}^{\alpha}y_{\lambda_2}(x)\right|_{x=0}=0 \tag{28}$$

$$d_1 y_{\lambda_2}(\pi)+d_2 I_{\pi,-}^{1-\alpha}h(\pi)^C\,D_{0,+}^{\alpha}y_{\lambda_2}(\pi)=0 \tag{29}$$

we multiply equation (24) by function y_{λ_2} and (27) by function y_{λ_1} respectively and subtract:

$$
\left(\lambda_1-\lambda_2\right)w_\alpha(x)\,y_{\lambda_1}y_{\lambda_2}=y_{\lambda_1}L_{[\alpha,p,q]}y_{\lambda_2}-y_{\lambda_2}L_{[\alpha,p,q]}y_{\lambda_1}.
$$

Integrating over interval $[0,\pi]$ and applying relation (17) we note that the right-hand side of the integrated equality contains only boundary terms:

$$\begin{aligned}
(\lambda_1-\lambda_2)\int\limits_0^\pi w_\alpha(x)\,y_{\lambda_1}(x)\,y_{\lambda_2}(x)\,dx &= \int\limits_0^\pi y_{\lambda_1}L_{[\alpha,p,q]}y_{\lambda_2}(x)\,dx-\int\limits_0^\pi y_{\lambda_2}(x)\,L_{[\alpha,p,q]}y_{\lambda_1}(x)\,dx \\
&= \int\limits_0^\pi 2\lambda_2 p(x)\,y_{\lambda_1}(x)\,y_{\lambda_2}(x)\,dx-\int\limits_0^\pi 2\lambda_1 p(x)\,y_{\lambda_1}(x)\,y_{\lambda_2}(x)\,dx \\
-y_{\lambda_1}(x)\,I_{\pi,-}^{1-\alpha}h(x)^C D_{0,+}^{\alpha}y_{\lambda_2}(x) &\mid_0^\pi + y_{\lambda_2}(x)\,I_{\pi,-}^{1-\alpha}h(x)^C D_{0,+}^{\alpha}y_{\lambda_1}(x)\mid_0^\pi \\
(\lambda_1-\lambda_2)\int\limits_0^\pi (w_\alpha(x)+2p(x))\,y_{\lambda_1}(x)\,y_{\lambda_2}(x)\,dx &= -y_{\lambda_1}(x)\,I_{\pi,-}^{1-\alpha}h(x)^C D_{0,+}^{\alpha}y_{\lambda_2}(x)\mid_0^\pi \\
&\quad +y_{\lambda_2}(x)\,I_{\pi,-}^{1-\alpha}h(x)^C D_{0,+}^{\alpha}y_{\lambda_1}(x)\mid_0^\pi .
\end{aligned}$$

Using the boundary conditions $(25),(26),(28),(29)$, we find that

$$(\lambda_1-\lambda_2)\int\limits_0^\pi (w_\alpha(x)+2p(x))\,y_{\lambda_1}(x)\,y_{\lambda_2}(x)\,dx=0$$

where $\lambda_1\neq\lambda_2$.

Let us now give certain auxiliary functions. Because we use the functions, the first of them is as follows

$$I_{0,+}^{\alpha}\frac{(b-x)^{\alpha-1}}{\Gamma(\alpha)}=(\pi-0)^{\alpha-1}(x-0)^{\alpha}\ {}_1\Psi_2\left[\begin{array}{cc}(1,1) & \\ (\alpha,-1) & (\alpha+1,1)\end{array}\middle|-\frac{x-0}{\pi-0}\right]$$

where ${}_1\Psi_2$ is the Fox-Wright function [13].

$${}_1\Psi_2\left[\begin{array}{cc}(a_1,\alpha_1) & \\ (b_1,\beta_1) & (b_2,\beta_2)\end{array}\middle|z\right]=\sum_{k=0}^{\infty}\frac{\Gamma(a_1+\alpha_1 k)}{\Gamma(b_1+\beta_1 k)\,\Gamma(b_2+\beta_2 k)}\frac{z^k}{k!},$$

the properties of the function are determined by the parameters

$$\begin{aligned}
\Delta &= \beta_1+\beta_2-\alpha_1=-1 \\
\delta &= |\alpha_1|^{-\alpha_1}|\beta_1|^{\beta_1}|\beta_2|^{\beta_2}=1 \\
\mu &= b_1+b_2-\alpha_1+\frac{1-2}{2}=2\alpha-\frac{1}{2},
\end{aligned}\tag{30}$$

considering Theorem 4, we note that this function is continuous in $[0,\pi]$ when order $\alpha>1/2$, i.e. $\mu>1/2$. For $0<\alpha\leq 1/2$ ıt is discontinuous at end $x=\pi$. The explicitly calculated function allows to estimate the second component of stationary function ϕ_0 of the differential part of diffusion operator

$$D_{\pi,-}^{\alpha}h(x)^C D_{0,+}^{\alpha}\phi_0(x)=0$$

which looks as follows

$$\phi_0(x)=\xi_1+\xi_2 I_{0,+}^{\alpha}\frac{(\pi-x)^{\alpha-1}}{\Gamma(\alpha)\,h(x)}=\xi_1+\xi_2\psi(\alpha,0,x)\,. \tag{31}$$

The next function is the following integral

$$\varphi(x) = I^{\alpha}_{0,+}I^{\alpha}_{\pi,-}1 = I^{\alpha}_{0,+}\frac{(\pi-x)^{\alpha}}{\Gamma(\alpha+1)} = (\pi-0)^{\alpha}(x-0)^{\alpha} \quad \times_1\Psi_2\left[\begin{array}{cc} (1,1) & \\ (\alpha+1,-1) & (\alpha+1,1) \end{array}\middle| -\frac{x-0}{\pi-0}\right] \tag{32}$$

Again, using the Theorem 4 and calculating parameters according to (30)

$$\Delta = -1, \quad \delta = 1, \quad \mu = 2\alpha + \frac{1}{2}$$

we conclude

$$\alpha > 0 \implies \mu > \frac{1}{2}$$

and the obtained Fox-Wright function (32) is continuous in interval $[0,\pi]$ for any positive order α.

Lemma 12. Let $\alpha > 1/2$ and denote

$$Y_{\lambda}(y) = (2\lambda p(x) + q(x))\, y_{\lambda}(x) + \lambda w_{\alpha} y_{\lambda}(x)$$

$$\tilde{\Delta} = c_1 d_2 + c_2 d_1 + c_1 d_1 \psi(\alpha, 0, \pi)$$

Assume $\tilde{\Delta} \neq 0$. Then, on the $C[a,b]$ -space, the regular $(12-14)$ is equivalent to the integral equation

$$y_{\lambda}(x) = -I^{\alpha}_{0,+}\frac{1}{h(x)}I^{\alpha}_{\pi,-}Y_{\lambda}(y) + A(x)\int_0^{\pi} Y_{\lambda}(y)\,dx + B(x)\left(I^{\alpha}_{0,+}\frac{1}{h(x)}I^{\alpha}_{\pi,-}Y_{\lambda}(y)\right)|_{x=\pi} \tag{33}$$

where the coefficents $A(x)$ and $B(x)$ are

$$A(x) = \frac{c_2}{\tilde{\Delta}}\left[-d_2 + d_1\psi(\alpha,0,x) - \psi(\alpha,0,\pi)\right]$$

$$B(x) = \frac{d_1}{\tilde{\Delta}}\left[c_1\psi(\alpha,0,x) + c_2\right]$$

and functions ψ is defined in (31).

proof. By aid of composition rules, equation (12) can be rewriten as follows:

$$D^{\alpha}_{\pi,-}h(x)^{C}D^{\alpha}_{0,+}\left[y_{\lambda}(x) + I^{\alpha}_{0,+}\frac{1}{h(x)}I^{\alpha}_{\pi,-}Y_{\lambda}(y)\right] = 0.$$

The last equality refers that on the $C[a,b]$ -space the function in brackets is a stationary function of fractional differential operator (FSLPDO). $D^{\alpha}_{\pi,-}h(x)^{C}D^{\alpha}_{0,+}$ which according to (31) can be found as

$$\phi_0 = \xi_1 + \xi_2 I^{\alpha}_{0,+}\frac{(\pi-x)^{\alpha-1}}{\Gamma(\alpha)h(x)} = \xi_1 + \xi_2\psi(\alpha,0,x)$$

equation (12) in the form of

$$y_{\lambda}(x) + I^{\alpha}_{0,+}\frac{1}{h(x)}I^{\alpha}_{\pi,-}Y_{\lambda}(y) = \xi_1 + \xi_2\psi(\alpha,0,x), \tag{34}$$

to end the proof we should connect coefficients ξ_j, $j=1,2$ values c_j,d_j $j=1,2$ determining the boundary conditions $(13-14)$. Let us note that the following formula results from composition rules (9) and equation (34)

$$I^{1-\alpha}_{\pi,-}h(x)^{C}D^{\alpha}_{0,+}y_{\lambda}(x) = -I^{1}_{\pi,-}Y_{\lambda}(y) + \xi_2,$$

for continuous function y_λ, we obtain the following values a the ends

$$I_{\pi,-}^{1-\alpha} h(x)^C D_{0,+}^{\alpha} y_\lambda(x)\Big|_{x=0} = -\int_0^\pi Y_\lambda(y) + \xi_2 \tag{35}$$

$$I_{\pi,-}^{1-\alpha} h(x)^C D_{0,+}^{\alpha} y_\lambda(x)\Big|_{x=\pi} = \xi_2 \tag{36}$$

respectively for y_λ, using (34) we find

$$y_\lambda(0) = \phi_0(0) = \xi_1 \tag{37}$$

$$\begin{aligned} y_\lambda(\pi) &= \phi_0(\pi) - I_{0,+}^{\alpha} \frac{1}{h(x)} I_{\pi,-}^{\alpha} Y_\lambda(y)\Big|_{x=\pi} \\ &= \xi_1 + \xi_2 \psi(\alpha, 0, \pi) - I_{0,+}^{\alpha} \frac{1}{h(x)} I_{\pi,-}^{\alpha} Y_\lambda(y)\Big|_{x=\pi}. \end{aligned} \tag{38}$$

The following set of linear equations for coefficients ξ_j results from $(35-39)$

$$c_2 X = c_2\xi_2 - c_1\xi_1 \tag{39}$$

$$d_1\xi_1 + \xi_2\left(d_2 + d_1\psi(\alpha,0,\pi)\right) = d_1 F$$

where $X = \int\limits_0^\pi Y_\lambda(y)\,dx$ and $F = I_{0,+}^{\alpha}\frac{1}{h(x)} I_{\pi,-}^{\alpha} Y_\lambda(y)\Big|_{x=\pi}$.

Since $\tilde{\Delta} \neq 0$, the solution for coefficients ξ_j $(j = 1, 2)$ is unique:

$$\xi_1 = \left(F c_2 d_1 - c_2 X\left(d_2 + d_1\psi(\alpha,0,\pi)\right)\right)/\tilde{\Delta}$$

$$\xi_2 = \left(c_1 d_1 F + c_2 d_1 X\right)/\tilde{\Delta}$$

substituting the above solution into (34) we recover the equivalent integral equation (33).

Furthermore we give notation such as

$$A = \|A(x)\|, \quad m_p = \min_{x\in[0,\pi]} |p(x)|,$$

$$B = \|B(x)\|, \quad M_\varphi = \|\varphi(x)\|,$$

where $\|.\|$ denotes the supremum norm on space $C[0,\pi]$.

Teorem 13. Let $\alpha > 1/2$. Suppose that $\tilde{\Delta} \neq 0$. Then unique continuous eigenfunction y_λ for regular fractional Sturm-Liouville problem with diffusion operator $(12-14)$ corresponding to each eigenvalue obeying

$$\|(2\lambda p(x) + q(x)) + \lambda w_\alpha\| < \frac{m_p}{M_\varphi + B\varphi(\pi) + A(\pi)\, m_p} \tag{40}$$

exists and such eigenvalue is simple.

Proof. We have to say that equation (33) can be interpreted as a fixed point condition on function space $C[0,\pi]$,

$$y_\lambda = T y_\lambda$$

where the mapping on the right-hand side is for any continuous function $g \in C[0,\pi]$ defined as

$$Tg = -I_{0,+}^{\alpha}\frac{1}{h(x)}I_{\pi,-}^{\alpha}Y_{\lambda}(y) + A(x)\int_{0}^{\pi} Y_{\lambda}(y)\,dx + B(x)\left(I_{0,+}^{\alpha}\frac{1}{h(x)}I_{\pi,-}^{\alpha}Y_{\lambda}(y)\right)|_{x=\pi}$$

The following inequality will be useful in further estimations:

$$\begin{aligned}
\|Y_{\lambda}(g) - Y_{\lambda}(r)\| &= \|((2\lambda p(x)+q)\,g + \lambda w_{\alpha} g) - ((2\lambda p(x)+q)\,r + \lambda w_{\alpha} r)\| \\
&= \|(2\lambda p(x)+q)(g-r) + \lambda w_{\alpha}(g-r)\| = \|(g-r)((2\lambda p(x)+q)+\lambda w_{\alpha})\| \\
&\le \|g-r\|\,\|(2\lambda p(x)+q)+\lambda w_{\alpha}\|\,.
\end{aligned}$$

By performing necassary operations for distance between images Tg and Tr for a pair of arbitrary continuous functions $g, r \in C[0,\pi]$.

$$\begin{aligned}
\|Tg - Tr\| &= \Bigg\|\left[-I_{0,+}^{\alpha}\frac{1}{h(x)}I_{\pi,-}^{\alpha}Y_{\lambda}(g) + A(x)\int_{0}^{\pi} Y_{\lambda}(g)\,dx + B(x)\left(I_{0,+}^{\alpha}\frac{1}{h(x)}I_{\pi,-}^{\alpha}Y_{\lambda}(g)\right)|_{x=\pi}\right] - \\
&\quad \left[-I_{0,+}^{\alpha}\frac{1}{h(x)}I_{\pi,-}^{\alpha}Y_{\lambda}(r) + A(x)\int_{0}^{\pi} Y_{\lambda}(r)\,dx + B(x)\left(I_{0,+}^{\alpha}\frac{1}{h(x)}I_{\pi,-}^{\alpha}Y_{\lambda}(r)\right)|_{x=\pi}\right]\Bigg\| \\
&= \Bigg\|-I_{0,+}^{\alpha}\frac{1}{h(x)}I_{\pi,-}^{\alpha}\left(Y_{\lambda}(g) - Y_{\lambda}(r)\right) + A(x)\int_{0}^{\pi}\left(Y_{\lambda}(g) - Y_{\lambda}(r)\right)dx + B(x) \\
&\quad \left(I_{0,+}^{\alpha}\frac{1}{h(x)}I_{\pi,-}^{\alpha}\left(Y_{\lambda}(g) - Y_{\lambda}(r)\right)\right)\Bigg|_{x=\pi}\Bigg\| \\
&\le \|g-r\|\,.\,\|(2\lambda p(x)+q)+\lambda w_{\alpha}\|\left(\frac{M_{\varphi}}{m_p} + A\pi + \frac{B\varphi(\pi)}{m_p}\right) \\
&\le \|g-r\|\,L
\end{aligned}$$

where constant $L = \|(2\lambda p(x)+q)+\lambda w_{\alpha}\|\left(\frac{M_{\varphi}}{m_p} + A\pi + \frac{B\varphi(\pi)}{m_p}\right)$. By means of (40), we state that mapping T is a contraction on space $\langle C[0,\pi], \|.\|\rangle$

$$\|Tg - Tr\| \le \|g-r\|\,L \quad L \in (0,1)\,.$$

Hence, a unique fixed point enounced as $y_{\lambda} \in C[0,\pi]$ exists that solves equation $(12, 33)$ and satisfies the boundary conditions $(13, 14)$, provided (40) is applied. In that case such eigenvalues are simple. The proof is completed.

Conclusion

In this article, we have extended the scope of some properties of fractional Sturm-Liouville problem for the diffusion operator. We pointed that its eigenvalues related to the fractional Sturm Liouville problem for diffusion operator with the certain boundary conditions are real and its eigenfunctions corresponding to distinct eigenvalues are ortohogonal. Furthermore, we showed that the operator is self adjoint. These properties are similar to those for integer Sturm Liouville problem of diffusion operator. Our new results are original and important for the fractional Sturm-Liouville theory.

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